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2003 J. Phys. A: Math. Gen. 36 8025

(http://iopscience.iop.org/0305-4470/36/29/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8025–8034 PII: S0305-4470(03)62948-3

Rational approximate symmetries of KdV equation

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Received 1 May 2003, in final form 16 June 2003
Published 8 July 2003
Online at stacks.iop.org/JPhysA/36/8025

Abstract
We construct one-parameter deformation of the Dorfman Hamiltonian operator for the Riemann hierarchy using the quasi-Miura transformation from topological field theory. In this way, one can get the rational approximate symmetries of the KdV equation and then investigate its bi-Hamiltonian structure.

PACS numbers: 02.30.Ik, 02.30.Mv, 11.10.Ef

1. Introduction

In this paper, we will investigate the one-parameter deformation of the Dorfman Hamiltonian operator ($D = \partial_x$)

$$J = D\frac{1}{v_x}D\frac{1}{v_x}D \tag{1}$$

which is of third order and compatible with the differential operator D, i.e., $J + \lambda D$ is a Hamiltonian operator for any λ [3]. The deformation of the bi-Hamiltonian pair J and D satisfies the Jacobi identity only up to a certain order of the parameter of the deformation. The problem is how we can find the deformation such that the bi-Hamiltonian structure can be preserved. One way to construct this deformation is borrowed from the free energy of the topological field theory (TFT) [5] (and references therein). The free energy satisfies the universal loop equation (p 157 in [5]). From the free energy, one can construct the so-called quasi-Miura transformation to get the deformation (see below).

From the deformation of the bi-Hamiltonian pair, one can also get the deformation of the recursion operator JD^{-1} to the genus-one correction (ϵ^2-correction). The deformed recursion operator can be used to generate higher-order symmetries, which commute with each other only up to $O(\epsilon^4)$. In doing so, we can deform the original integrable system to include ϵ^2-correction . The rational approximate symmetries of the KdV equation are established using the method.

Let us start with the well-known Riemann equation

$$v_t = vv_x. \tag{2}$$

It is also called the dispersionless KdV (dKdV) equation. The integrability of (2) is that it has an infinite sequence of commuting Hamiltonian flows ($t_1 = t$)

$$v_{t_n} = v^n v_x \qquad n = 1, 2, 3, \ldots. \tag{3}$$

The Riemann hierarchy (3) has the bi-Hamiltonian structure [11]

$$v_{t_n} = \frac{1}{(n+1)(n+2)} D\delta H_{n+2} = \frac{1}{(n+1)(n+2)(n+3)(n+4)} J\delta H_{n+4} \tag{4}$$

where $H_n = \int v^n \,\mathrm{d}x$, δ is the variational derivative and J is the Dorfman Hamiltonian operator (1). From the bi-Hamiltonian structure (4), the recursion operator is defined as

$$L = JD^{-1} = D\frac{1}{v_x}D\frac{1}{v_x} = R^2 \tag{5}$$

where

$$R = D\frac{1}{v_x} \tag{6}$$

is the Olver–Nutku recursion operator [11], i.e., the square root of the recursion operator L. One can easily check that R (or L) satisfies the following recursion operator equation associated with the Riemann equation (2)

$$A_t = [v_x + vD, A] \tag{7}$$

where A is a (pseudo-)differential operator. Then from the recursion operator theory [10], one can establish new symmetries of (2) by the Olver–Nutku recursion operator (6) repeatedly

$$v_{\tau_n} = R^n 1 \qquad n = 1, 2, 3, \ldots. \tag{8}$$

The new symmetries (8) of (2), i.e.,

$$(v_t)_{\tau_n} = \left(v_{\tau_n}\right)_t$$

will correspond to the 'superintegrability' of the Riemann equation (2) [14].

Next, to deform the recursion operator (6), we use the free energy in TFT of the famous KdV equation

$$u_t = uu_x + \frac{\epsilon^2}{12}u_{xxx} \tag{9}$$

to construct the quasi-Miura transformation as follows. The free energy F of KdV equation (9) in TFT has the form $\left(F_0 = \frac{1}{6}v^3\right)$

$$F = \frac{1}{6}v^3 + \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)}).$$

Let

$$\begin{aligned}\triangle F &= \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)}) \\ &= F_1(v; v_x) + \epsilon^2 F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}) \\ &\quad + \epsilon^4 F_3(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}, \ldots, v^{(7)}) + \cdots.\end{aligned}$$

The $\triangle F$ will satisfy the loop equation (p 151 in [5])

$$\sum_{r\geqslant 0}\frac{\partial\triangle F}{\partial v^{(r)}}\partial_x^r\frac{1}{v-\lambda}+\sum_{r\geqslant 1}\frac{\partial\triangle F}{\partial v^{(r)}}\sum_{k=1}^{r}\binom{r}{k}\partial_x^{k-1}\frac{1}{\sqrt{v-\lambda}}\partial_x^{r-k+1}\frac{1}{\sqrt{v-\lambda}}$$
$$=\frac{1}{16\lambda^2}-\frac{1}{16(v-\lambda)^2}-\frac{\kappa_0}{\lambda^2}+\frac{\epsilon^2}{2}\sum_{k,l\geqslant 0}\left[\frac{\partial^2\triangle F}{\partial v^{(k)}\partial v^{(l)}}+\frac{\partial\triangle F}{\partial v^{(k)}}\frac{\partial\triangle F}{\partial v^{(l)}}\right]$$
$$\times\,\partial_x^{k+1}\frac{1}{\sqrt{v-\lambda}}\partial_x^{l+1}\frac{1}{\sqrt{v-\lambda}}-\frac{\epsilon^2}{16}\sum_{k\geqslant 0}\frac{\partial\triangle F}{\partial v^{(k)}}\partial_x^{k+2}\frac{1}{(v-\lambda)^2}. \tag{10}$$

Then we can determine $F_1, F_2, F_3, \ldots$ recursively by substituting $\triangle F$ into equation (10). For F_1, one obtains

$$\frac{1}{v-\lambda}\frac{\partial F_1}{\partial v}-\frac{3}{2}\frac{v_x}{(v-\lambda)^2}\frac{\partial F_1}{\partial v_x}=\frac{1}{16\lambda^2}-\frac{1}{16(v-\lambda)^2}-\frac{\kappa_0}{\lambda^2}.$$

From this, we have

$$\kappa_0=\tfrac{1}{16}\qquad F_1=\tfrac{1}{24}\log v_x.$$

The next term $F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx})$ is presented in the appendix. Now, one can define the quasi-Miura transformation as

$$u=v+\epsilon^2(\triangle F)_{xx}=v+\epsilon^2(F_1)_{xx}+\epsilon^4(F_2)_{xx}+\cdots$$
$$=v+\frac{\epsilon^2}{24}(\log v_x)_{xx}+\epsilon^4\left(\frac{v_{xxxx}}{1152v_x^2}-\frac{7v_{xx}v_{xxx}}{1920v_x^3}+\frac{v_{xx}^3}{360v_x^4}\right)_{xx}+\cdots. \tag{11}$$

One remarks that a Miura-type transformation means the coefficients of ϵ are homogeneous polynomials in the derivatives $v_x, v_{xx}, \ldots, v^{(m)}$ (p 37 in [5], [7]) and 'quasi' means those of ϵ are quasi-homogeneous rational fuctions in the derivatives also (p 109 in [5] and see also [13]).

The truncated quasi-Miura transformation

$$u=v+\sum_{n=1}^{g}\epsilon^{2n}[F_n(v; v_x, v_{xx}, \ldots, v^{(3g-2)})]_{xx} \tag{12}$$

has the basic property (p 117 in [5]) that it reduces the Magri Poisson pencil of the KdV equation (9) [8]

$$\{u(x), u(y)\}_\lambda=[u(x)-\lambda]D\delta(x-y)+\tfrac{1}{2}u_x(x)\delta(x-y)+\frac{\epsilon^2}{8}D^3\delta(x-y) \tag{13}$$

to the Poisson pencil of the Riemann hierarchy (3):

$$\{v(x), v(y)\}_\lambda=[v(x)-\lambda]D\delta(x-y)+\tfrac{1}{2}v_x(x)\delta(x-y)+O(\epsilon^{2g+2}). \tag{14}$$

One can also say that the truncated quasi-Miura transformation (12) deforms the KdV equation (9) to the Riemann equation (2) up to $O(\epsilon^{2g+2})$. And conversely, we can also think that the Poisson pencil (14) for the Riemann hierarchy is deformed to get the Magri Poisson pencil (13) of genus-g correction after the truncated quasi-Miura transformation (12). So a very natural question arises: under the truncated quasi-Miura transformation (12), is the deformed Dorfman Hamiltonian operator $J(\epsilon)$ of (1) still Hamiltonian and compatible with D up to $O(\epsilon^{2g+2})$? The answer is affirmative for $g = 1$, i.e.,

$$u = v + \frac{\epsilon^2}{24}(\log v_x)_{xx} + O(\epsilon^4) \tag{15}$$

or

$$v = u - \frac{\epsilon^2}{24}(\log u_x)_{xx} + O(\epsilon^4) \tag{16}$$

and it is the main purpose of this paper.

Also, from the deformed recursion operator $R(\epsilon)$ of the Olver–Nutku recursion operator (6), we can generate rational approximate symmetries of the KdV equation (9) up to $O(\epsilon^4)$. These symmetries are different from those generated by the Magri Poisson pencil (13). Then one can call them the 'superintegrability' of the KdV equation.

Finally, one remarks that in general integrable dispersive deformation for integrable dispersionless systems is not unique [1, 6, 9, 13]. For deformations of bi-Hamiltonian PDEs of hydrodynamic type with one dependent variable, we refer to [7].

The paper is organized as follows. In the next section, we construct the genus-one deformation of Olver–Nutku recursion operator. In section 3, the bi-Hamiltonian structure of the rational approximate symmetries of KdV equation (9) is investigated. In the final section, we discuss some problems to be investigated.

2. Quasi-Miura transformation of the Olver–Nutku recursion operator

In this section, we will investigate the Hamiltonian operator D and the Olver–Nutku recursion operator (6) under the truncated quasi-Miura transformation (12) for $g = 1$.

In the new 'u-coordinate', D and R will be given by the operator

$$D(\epsilon) = M^* D M \tag{17}$$
$$R(\epsilon) = M^* R (M^*)^{-1} \tag{18}$$

where

$$M = 1 - \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2$$
$$M^* = 1 + \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D$$

M^* being the adjoint operator of M. Then using (17), (18) and (16), we can yield, after a simple calculation,

$$D(\epsilon) = D + O(\epsilon^4)$$
$$\begin{aligned} R(\epsilon) &= D\frac{1}{u_x} + \frac{\epsilon^2}{24} D \left(D\frac{1}{u_x}D^2\frac{1}{u_x} - \frac{1}{u_x}D^2\frac{1}{u_x}D + \frac{(\log u_x)_{xxx}}{u_x^2} \right) + O(\epsilon^4) \\ &= D\frac{1}{u_x} + \frac{\epsilon^2}{12} D \left[\frac{1}{u_x}\left(\frac{1}{u_x}\right)_x D^2 + \left(\frac{1}{u_x}\left(\frac{1}{u_x}\right)_x\right)_x D - 3u_x^{-5}u_{xx}^3 \right. \\ &\quad \left. + 2u_x^{-4}u_{xx}u_{xxx} \right] + O(\epsilon^4). \end{aligned}$$

We hope that $R(\epsilon)$ is a recursion operator of the KdV equation (9). Indeed, it is the following

Theorem 1. *$R(\epsilon)$ satisfies the recursion operator equation of KdV equation*

$$R(\epsilon)_t = \left[u_x + uD + \frac{\epsilon^2}{12}D^3, R(\epsilon)\right] + O(\epsilon^4). \tag{19}$$

Proof. Direct calculations. □

We can think of (19) as the genus-one deformation of (7). One remarks that the recursion operator

$$u + \frac{u_x}{2}D^{-1} + \frac{\epsilon^2}{8}D^2$$

of Magri pencil (13) will also satisfy the recursion operator equation (19) but there is no higher-order correction. Moreover, we know that in general the recursion operator is non-local [10] and hence the local property of $R(\epsilon)$ is special from this point of view.

Now from theorem 1, one will construct infinite rational symmetries (up to $O(\epsilon^4)$) of KdV equation (9) using recursion operator $R(\epsilon)$ as follows:

$$u_{\tau_n} = R^n(\epsilon)1 + O(\epsilon^4) \qquad n = 1, 2, 3, \ldots \tag{20}$$

which is the genus-one deformation of (8). For example,

$$\begin{aligned} u_{\tau_1} &= R(\epsilon)1 = \left[\frac{1}{u_x} + \frac{\epsilon^2}{12}\left(-3u_x^{-5}u_{xx}^3 + 2u_x^{-4}u_{xx}u_{xxx}\right)\right]_x + O(\epsilon^4) \\ u_{\tau_2} &= R^2(\epsilon)1 = \left\{\frac{1}{u_x}\left(\frac{1}{u_x}\right)_x + \frac{\epsilon^2}{12}\left[30u_x^{-7}u_{xx}^4 - 30u_x^{-6}u_{xx}^2u_{xxx}\right.\right. \\ &\qquad \left.\left. + 3u_x^{-5}u_{xxx}^2 + 3u_x^{-5}u_{xx}u_{xxxx}\right]\right\}_x + O(\epsilon^4). \end{aligned} \tag{21}$$

Also, we note that one can also obtain (20) by (15) as follows. Since

$$u_{\tau_{n+1}} = v_{\tau_{n+1}} + \frac{\epsilon^2}{24}\left(\frac{v_{\tau_n x}}{v_x}\right)_{xx} + O(\epsilon^4)$$

using (16), after some calculations, we can obtain

$$\begin{aligned} u_{\tau_{n+1}} &= \left\{\frac{u_{\tau_n}}{u_x} + \frac{\epsilon^2}{24}\left[\frac{(\log u_x)_{xxx}}{u_x^2}u_{\tau_n}\right] + \frac{\epsilon^2}{24}\left[\left(\frac{u_{\tau_n}}{u_x}\right)_{xx}\Big/ u_x\right]_x \right. \\ &\quad \left. - \frac{\epsilon^2}{24}\left[\left(\frac{u_{\tau_n x}}{u_x}\right)_{xx}\Big/ u_x\right]\right\}_x + O(\epsilon^4) \\ &= R(\epsilon)u_{\tau_n} + O(\epsilon^4). \end{aligned}$$

3. Bi-Hamiltonian structure of rational approximate symmetries

In this section, we will prove the bi-Hamiltonian structure of (20) for even flows, i.e., $n = 2k, k \geqslant 1$.

Firstly, the deformed Dorfman Hamiltonian operator $J(\epsilon)$ under the quasi-Miura transformation (15) is

$$\begin{aligned} J(\epsilon) &= R^2(\epsilon)D(\epsilon) \\ &= D\frac{1}{u_x}D\frac{1}{u_x}D + \frac{\epsilon^2}{24}D\left[\frac{1}{u_x}D\frac{(\log u_x)_{xxx}}{u_x^2} + \frac{(\log u_x)_{xxx}}{u_x^2}D\frac{1}{u_x}\right. \\ &\quad \left. + D\frac{1}{u_x}D^2\frac{1}{u_x}D\frac{1}{u_x} - \frac{1}{u_x}D\frac{1}{u_x}D^2\frac{1}{u_x}D\right]D + O(\epsilon^4). \end{aligned} \tag{22}$$

Then we have the following.

Theorem 2. *(i) $J(\epsilon)$ is a Hamiltonian operator up to $O(\epsilon^4)$. (ii) $J(\epsilon)$ and $D(\epsilon)$ form a bi-Hamiltonian pair up to $O(\epsilon^4)$.*

Proof. (i) The skew-symmetric property of operator (22) is obvious. To prove $J(\epsilon)$ is a Hamiltonian operator, we must verify that $J(\epsilon)$ satisfies the Jacobi identities up to $O(\epsilon^4)$. Following [10, 11], we introduce the arbitrary basis of tangent vector Θ, which is then conveniently manipulated according to the rules of exterior calculus. The Jacobi identities are given by the compact expression

$$P(\epsilon) \wedge \delta I = O(\epsilon^4)(\text{mod. div.}) \tag{23}$$

where $P(\epsilon) = J(\epsilon)\Theta$, $I = \frac{1}{2}\Theta \wedge P(\epsilon)$ and δ denotes the variational derivative. The vanishing of the tri-vector (23) modulo a divergence is equivalent to the satisfaction of Jacobi identities.

Now, a lengthy and tedious calculation can yield

$$P(\epsilon) = \left\{\frac{1}{u_x}\left(\frac{\Theta_x}{u_x}\right)_x + \frac{\epsilon^2}{24}\left[\frac{1}{u_x}\left(\frac{(\log u_x)_{xxx}}{u_x^2}\Theta_x\right)_x + \frac{(\log u_x)_{xxx}}{u_x^2}\left(\frac{\Theta_x}{u_x}\right)_x \right.\right.$$
$$\left.\left. + \left(\frac{1}{u_x}\left(\frac{1}{u_x}\left(\frac{\Theta_x}{u_x}\right)_x\right)_{xx}\right)_x - \frac{1}{u_x}\left(\frac{1}{u_x}\left(\frac{\Theta_{xx}}{u_x}\right)_{xx}\right)_x\right]\right\}_x + O(\epsilon^4)$$

and

$$I = \frac{1}{2}\Theta \wedge P(\epsilon) = -\frac{1}{2u_x^2}\Theta_x \wedge \Theta_{xx} + \frac{\epsilon^2}{24}\left\{-5u_x^{-6}u_{xx}^3\Theta_x \wedge \Theta_{xx} + 3u_x^{-5}u_{xx}^2\Theta_x \wedge \Theta_{xxx}\right.$$
$$\left. - 2u_x^{-4}u_{xx}\Theta_{xx} \wedge \Theta_{xxx}\right\} + O(\epsilon^4) \quad (\text{mod. div.}).$$

Then

$$\delta I = \left(3u_x^{-4}u_{xx}\Theta_x \wedge \Theta_{xx} - u_x^{-3}\Theta_x \wedge \Theta_{xxx}\right) + \frac{\epsilon^2}{24}\left\{60u_x^{-7}u_{xx}^3\Theta_x \wedge \Theta_{xx}\right.$$
$$- 30u_x^{-6}u_{xx}u_{xxx}\Theta_x \wedge \Theta_{xx} - 30u_x^{-6}u_{xx}^2\Theta_x \wedge \Theta_{xxx}$$
$$+ 6u_x^{-5}u_{xxx}\Theta_x \wedge \Theta_{xxx} + 6u_x^{-5}u_{xx}\Theta_{xx} \wedge \Theta_{xxx}$$
$$\left. + 6u_x^{-5}u_{xx}\Theta_x \wedge \Theta_{xxxx} - 2u_x^{-4}\Theta_{xx} \wedge \Theta_{xxxx}\right\}_x + O(\epsilon^4). \tag{24}$$

Finally,

$$P(\epsilon) \wedge \delta I = 0 - \frac{\epsilon^2}{24}\left\{-7u_x^{-8}u_{xx}u_{xxx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxxx} + 7u_x^{-8}u_{xx}^2\Theta_x \wedge \Theta_{xxx} \wedge \Theta_{xxxx}\right.$$
$$- 7u_x^{-7}u_{xx}\Theta_{xx} \wedge \Theta_{xxx} \wedge \Theta_{xxxx} + \left(7u_x^{-8}u_{xx}u_{xxxx}\right.$$
$$\left. - u_x^{-7}u_{xxxxx}\right)\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxx} - u_x^{-7}u_{xx}\Theta_x \wedge \Theta_{xxx} \wedge \Theta_{xxxxx}$$
$$\left. + u_x^{-7}u_{xxx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxxxx} + u_x^{-6}\Theta_{xx} \wedge \Theta_{xxx} \wedge \Theta_{xxxxx}\right\}$$
$$= \left(u_x^{-6}\Theta_{xx} \wedge \Theta_{xxx} \wedge \Theta_{xxxx}\right)_x - \left(u_x^{-7}u_{xx}\Theta_x \wedge \Theta_{xxx} \wedge \Theta_{xxxx}\right)_x$$
$$+ \left(u_x^{-7}u_{xxx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxxx}\right)_x - \left(u_x^{-7}u_{xxxx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxx}\right)_x + O(\epsilon^4)$$

which is a total derivative so that the Jacobi identities are satisfied and this completes the proof of (i).

(ii) Since $J(\epsilon)$ and $D(\epsilon)$ are Hamiltonian operators, we need only verify the additional condition

$$P(\epsilon) \wedge \delta I_D + D(\Theta) \wedge \delta I = O(\epsilon^4)$$

where

$$I_D = \tfrac{1}{2}\Theta \wedge D(\Theta) = \tfrac{1}{2}\Theta \wedge \Theta_x$$

δI and $P(\epsilon)$ are defined in (23), modulo a divergence.

Obviously, $\delta I_D = O(\epsilon^4)$. So we will check $D(\Theta) \wedge \delta I = \Theta_x \wedge \delta I = O(\epsilon^4)$. From (24), we have

$$\begin{aligned}\Theta_x \wedge \delta I &= 0 + \frac{\epsilon^2}{24}\big\{6u_x^{-5}u_{xxx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxx} \\ &\quad - 30u_x^{-6}u_{xx}^2\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxx} + 6u_x^{-5}u_{xx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxxx}\big\} \\ &= \frac{\epsilon^2}{24}\big\{6u_x^{-5}u_{xx}\Theta_x \wedge \Theta_{xx} \wedge \Theta_{xxx}\big\}_x + O(\epsilon^4).\end{aligned}$$

This completes the proof of (ii). □

Remark. Although the quasi-Miura transformation (15) is of a change of coordinates (including derivatives), it is non-trivial to see that $J(\epsilon)$ is a Hamiltonian operator (up to $O(\epsilon^4)$). It is because the change of coordinates, in general, will not preserve the Jacobi identities.

Since $J(\epsilon)$ and $D(\epsilon)$ form a Hamiltonian pair, we will find the Hamiltonian densities of the even flows of the rational approximate symmetries of the KdV equation (9) up to $O(\epsilon^4)$

$$u_{\tau_{2n}} = R^{2n}(\epsilon)1 = D(\epsilon)\frac{\delta \tilde{H}_n(\epsilon)}{\delta u} = J(\epsilon)\frac{\delta \tilde{H}_{n-1}(\epsilon)}{\delta u} \qquad n = 1, 2, 3, \ldots \tag{25}$$

in the following way. Firstly, we note that from (8) we have

$$v_{\tau_{2n}} = R^{2n}1 = D(K_{2n+1}) = J(K_{2n-1}) \tag{26}$$

where

$$\begin{aligned}K_1 &= x \\ K_3 &= \frac{1}{v_x}\left(\frac{1}{v_x}\right)_x \\ K_5 &= \frac{1}{v_x}\left(\frac{1}{v_x}(K_3)_x\right)_x \\ &\vdots \\ K_{2n+1} &= \frac{1}{v_x}\left(\frac{1}{v_x}(K_{2n-1})_x\right)_x.\end{aligned}$$

From the bi-Hamiltonian structure of J and D, one can construct the Hamiltonian densities of (26) using the method described in [4]. Secondly, from the Hamiltonian structure of $J(\epsilon)$ and $D(\epsilon)$, one can also construct the Hamiltonian densities of (25) using the quasi-Miura transformation (15). For example,

$$\begin{aligned}v_{\tau_2} &= R^2 1 = D\left(\frac{1}{v_x}\left(\frac{1}{v_x}\right)_x\right) = D\frac{\delta \hat{H}_1}{\delta v} = J\frac{\delta \hat{H}_0}{\delta v} \\ v_{\tau_4} &= R^4 1 = D\left(\frac{1}{v_x}\left(\frac{1}{v_x}(K_3)_x\right)_x\right) = D\frac{\delta \hat{H}_2}{\delta v} = J\frac{\delta \hat{H}_1}{\delta v}\end{aligned}$$

where

$$\hat{H}_0 = \int xv\,\mathrm{d}x$$
$$\hat{H}_1 = \frac{1}{2}\int \frac{1}{v_x}\,\mathrm{d}x$$
$$\hat{H}_2 = -\frac{1}{2}\int v_{xx}^2 v_x^{-5}\,\mathrm{d}x.$$

Then after the quasi-Miura transformation (16), one can obtain

$$\hat{H}_0(\epsilon) = \int x\left(u - \frac{\epsilon^2}{24}(\log u_x)_{xx}\right)\mathrm{d}x + O(\epsilon^4)$$
$$\hat{H}_1(\epsilon) = \frac{1}{2}\int\left[\frac{1}{u_x} + \frac{\epsilon^2}{24}\big(2u_x^{-5}u_{xx}^3 - 3u_x^{-4}u_{xx}u_{xxx} + u_x^{-3}u_{xxxx}\big)\right]\mathrm{d}x + O(\epsilon^4)$$
$$\hat{H}_2(\epsilon) = -\frac{1}{2}\int\left[u_{xx}^2u_x^{-5} + \frac{\epsilon^2}{24}\big(22u_x^{-9}u_{xx}^5 - 39u_x^{-8}u_{xx}^3u_{xxx} + 13u_x^{-7}u_{xx}^2u_{xxxx} - 2u_x^{-6}u_{xx}u_{xxxxxx} + 6u_x^{-7}u_{xx}u_{xxx}^2\big)\right]\mathrm{d}x + O(\epsilon^4).$$

On the other hand, we can also verify using MAPLE that, noting (21),

$$u_{\tau_2} = R^2(\epsilon)1 = D(\epsilon)\frac{\delta\hat{H}_1(\epsilon)}{\delta u} = J(\epsilon)\frac{\delta\hat{H}_0(\epsilon)}{\delta u} + O(\epsilon^4)$$
$$u_{\tau_4} = R^4(\epsilon)1 = \left\{\frac{1}{u_x}\left(\frac{1}{u_x}\left(\frac{1}{u_x}\right)_x\right)_x + \frac{\epsilon^2}{12}\big[1050u_x^{-9}u_{xx}^3u_{xxxx} - 105u_x^{-8}u_{xxx}^3 + 3780u_x^{-11}u_{xx}^6 - 6300u_x^{-10}u_{xx}^4u_{xxx} + 2310u_x^{-9}u_{xx}^2u_{xxx}^2 - 420u_x^{-8}u_{xx}u_{xxx}u_{xxxx} + 5u_x^{-7}u_{xx}u_{xxxxxx} + 15u_x^{-7}u_{xxx}u_{xxxxx} + 10u_x^{-7}u_{xxxx}^2 - 105u_x^{-8}u_{xx}^2u_{xxxxx}\big]\right\}_x + O(\epsilon^4)$$
$$= D(\epsilon)\frac{\delta\hat{H}_2(\epsilon)}{\delta u} = J(\epsilon)\frac{\delta\hat{H}_1(\epsilon)}{\delta u}$$

which comes from the fact that the quasi-Miura transformation for $g = 1$ is canonical by theorem 2.

One remarks that the truncated τ_{2n}-flows are approximately integrable systems. We expect that solutions to such approximately integrable equations exhibit integrable behaviour at least for small physical parameters, for example, soliton solutions, as in [7]. But the truncated τ_{2n}-flows are very complicated and need further investigations.

4. Concluding remarks

We have studied the genus-one deformation of the Dorfman Hamiltonian operator using quasi-Miura transformation borrowed from the free energy of the topological field theory. Then one can prove that the deformed Hamiltonian operators $J(\epsilon)$ and $D(\epsilon)$ are still compatible and thus it provides the rational approximate symmetries of the KdV equation up to $O(\epsilon^4)$.

In spite of the results obtained, there are some interesting issues that deserve further investigation:

- We believe that theorems 1 and 2 can be generalized to higher genus, i.e., $g \geqslant 2$. However, the computations will become quite unmanageable.
- The Schwarzian KdV equation (degenerate Krichever–Novikov (KN) equation [12] or Ur–KdV equation [15]) is

$$v_t = v_{xxx} - \tfrac{3}{2} v_x^{-1} v_{xx}^2 = v_x \{v, x\}$$

where $\{v, x\}$ is the Schwarzian derivative. It is known that

$$\frac{1}{v_x} D \frac{1}{v_x}$$

and the Dorfman Hamiltonian operator J constitute a symplectic pair of the Schwarzian KdV equation [4]. Thus, under the quasi-Miura transformation we can also investigate the genus-one deformation of the Schwarzian KdV equation [2].
- One can generalize J to the polytropic gas system [11]. Using the universal loop equation of free energy (p 157 in [5]), we can also find the corresponding quasi-Miura transformations of two variables and study their deformations. Thus, the rational approximate symmetries of polytropic gas systems will also be obtained. But the computations are more involved and need further investigation.

Acknowledgments

The author is grateful to Professor Nutku for stimulating conversations on the Hamiltonian theory of polytropic gas systems. He would like to thank the unknown referees for their valuable suggestions. He also thanks the National Science Council for support under grant no NSC 91-2115-M-014-001.

Appendix

The equation for F_2 is

$$\frac{1}{(v-\lambda)^5}\left(\frac{105}{2048} v_x^2 - \frac{945}{16} v_x^4 \frac{\partial F_2}{\partial v_{xxxx}}\right) + \frac{1}{(v-\lambda)^4}\left(\frac{-49}{1536} v_{xx} + \frac{735}{8} v_x^2 v_{xx} \frac{\partial F_2}{\partial v_{xxxx}}\right.$$
$$\left. + \frac{105}{8} v_x^3 \frac{\partial F_2}{\partial v_{xxx}}\right) + \frac{1}{(v-\lambda)^3}\left[\frac{v_{xxx}}{192 v_x} - \frac{23 v_{xx}^2}{4608 v_x^2} - \left(16 v_{xx}^2 + \frac{87}{4} v_x v_{xxx}\right)\frac{\partial F_2}{\partial v_{xxxx}}\right.$$
$$\left. - \frac{55}{4} v_x v_{xx} \frac{\partial F_2}{\partial v_{xxx}} - \frac{15}{4} v_x^2 \frac{\partial F_2}{\partial v_{xx}}\right] + \frac{1}{(v-\lambda)^2}\left(3 v_{xxxx} \frac{\partial F_2}{\partial v_{xxxx}} + \frac{5}{2} v_{xxx} \frac{\partial F_2}{\partial v_{xxx}}\right.$$
$$\left. + 2 v_{xx} \frac{\partial F_2}{\partial v_{xx}} + \frac{3}{2} v_x \frac{\partial F_2}{\partial v_x}\right) - \frac{1}{(v-\lambda)} \frac{\partial F_2}{\partial v} = 0.$$

Let the coefficients of $\frac{1}{(v-\lambda)^i}$, $i = 1, 2, 3, 4, 5$, be equal to zero. Then one can obtain

$$F_2 = \frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4}.$$

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